

# Connectivity, persistence and fault diagnosis of interconnection networks based on $O_k$ and $2O_k$ graphs

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## *Abstract*

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Various fault-tolerance performance parameters including enumeration of openly disjoint paths, persistence, and resilience of odd graphs  $O_k$  and their doubly bipartite derivatives,  $2O_k$ , are analyzed in this paper. Using a remarkable partitioning property of these graphs, we propose an efficient semi-distributed fault-diagnosis scheme for these networks. It is shown through the performance parameters that both  $O_k$  and  $2O_k$  graphs have a high degree of fault tolerance and allow efficient fault diagnosis which make these networks comparable or even superior to some other well-known dense networks.

## 1. Introduction

Due to the growing interest in parallel processing a number of interconnection networks for multiprocessor systems have been proposed [1,20,22]. An interconnection network joins  $N$  nodes through bidirectional communication links. The nodes and the links can be represented, respectively, as vertices and edges of an undirected graph. A node is a combination of a processing unit and a message switching element. Most of these networks are designed to provide point-to-point communication among  $N$  processors operating in a multiple instruction multiple data (MIMD) environment. The number of processors ( $N$ ) for the future systems is expected to grow considerably large. For example, the teraflop system (TF-1) proposed by the Defense Advanced Research Project Agency (DARPA) is expected to use 32,000

general purpose processors with the design of interconnection being the major problem for its development [10].

An interconnection network plays an important role in the overall performance of a multiprocessor system. For such a network a number of topological parameters can be chosen which greatly effect the end-to-end delays, the fault-tolerance capability and the message routing complexity. The selection of these parameters mainly depends upon the designer of the network [7]. However, fault tolerance is becoming a major design problem in interconnection networks. This is due to the fact that with an increase in the size of system the failure of components (links/nodes) can be quite frequent. In order to continue supporting normal operation of a system during failures, it is highly desirable that the system should have a high degree of fault tolerance and under a limited number of failures the system's performance must remain acceptable. Also, a closely related but important design consideration for large multiprocessor interconnection is the ability of the system to diagnose itself and identify faulty components [2,17,19,21]. However, self-diagnosis is generally a computationally expensive procedure, as it consumes a considerable amount of system's time and requires a large number of resources for storing and processing test messages generated in the system. The amount of diagnosis overhead, such as the number of messages generated, greatly depends on the topological characteristics of the interconnection network and the nature of the diagnosis scheme. For example, a centralized diagnosis mechanism requires the least amount of overhead, but this scheme is slow since a single monitoring node needs to process all the test messages. Also, this scheme does not have any fault tolerance and is highly vulnerable to failure of the central monitoring node. On the other hand, a fully distributed approach is highly fault tolerant but it is very costly and the amount of traffic generated can be formidable. An efficient way to diagnose large multiprocessor systems is to use some sort of semi-distributed scheme.

In this paper, we analyze a number of fault-tolerance performance parameters of odd graphs ( $O_k$ ) and their doubly bipartite derivative ( $2O_k$ ), which have recently been proposed as possible graphs for designing interconnection networks [15]. We show that both  $O_k$  and  $2O_k$  graphs possess better fault-tolerance parameters than many other proposed architectures such as de Bruijn networks [20], cube-connected cycles [22], and flip-trees [20], etc. (Flip-tree is generated based upon a tree structure and interconnecting leaves of the tree according to some labeling scheme, see [20] for detail.) We also propose an efficient diagnosis scheme for these networks and show that the proposed scheme provides an improvement in the diagnosability which is of order of magnitude reported earlier [13,14]. The proposed scheme is semi-distributed and requires low overhead. The scheme is based on a remarkable property of  $O_k$  and  $2O_k$  networks which is their ability of partitioning into identical *spheres*. The partitioning is achieved by using a combinatorial structure called the *Hadamard matrix* [5,18].

In the next section we briefly describe some important properties of these graphs. The fault-tolerance capability and the related performance parameters are described

in Sections 3 and 4. Comparison with other networks are also given in Section 4. The partitioning property and a semi-distributed fault-diagnosis algorithm is given in Section 5. Finally, the concluding remarks are given in Section 6.

## 2. Some properties of $O_k$ and $2O_k$ graphs

The definitions of various graphical parameters used in this paper such as the degree  $k$ , the diameter  $d$ , regularity, the shortest graphical distance and path can be found in [16, Ch. 2].

An odd graph  $O_k$  is generated by taking  $k$ -subsets of the set  $V = \{0, 1, 2, \dots, 2k-2\}$  as vertices, and joining two vertices if their subsets have exactly one element in common [8]. Similarly, its bipartite doubling is the  $2O_k$  graph which is generated by taking  $k$  and  $(k-1)$ -subsets of  $V$  as vertices and joining two vertices by an edge if the subset associated with one vertex completely covers the other [15].

The code-theoretic representation of these graphs is quite straightforward. We can represent a  $k$ -subset of the set  $V$  as a binary codeword of length  $2k-1$ , having 1's in those positions which are in the subset. The number of 1's in a codeword is called its *weight* [18]. The *Hamming distance*, denoted as  $H_{xy}$ , between two binary codewords,  $x$  and  $y$ , is the number of positions at which these codewords differ. Therefore, in an  $O_k$  graph the vertices can also be represented as binary codewords of length  $2k-1$  and weight  $k$ . Two vertices  $x, y \in O_k$  are adjacent if  $H_{xy} = 2k-2$ . On the other hand, two vertices  $x, y \in 2O_k$  are adjacent if  $H_{xy} = 1$ .

Figure 1 shows  $O_3$  and  $2O_3$  graphs. The former is the well-known Petersen graph.

Some important properties of these graphs are as follows.

**Lemma 2.1.** *An  $O_k$  graph is regular with  $N = \binom{2k-1}{k}$  nodes, degree  $k$  and diameter  $d = k-1$  [8]. For a  $2O_k$  graph  $N = \binom{2k}{k}$ , degree is  $k$  and diameter  $d = 2k-1$  [15].*

**Theorem 2.2.** *Let  $\bar{y}$  be the complement of codeword  $y$ . Given two vertices  $x, y \in O_k$ , having the shortest graphical distance  $L_{xy}$  between them, the following relation holds:*

$$L_{xy} = \min(H_{xy}, H_{x\bar{y}}), \quad (1)$$

while for the  $2O_k$  graph

$$L_{xy} = H_{xy}. \quad (2)$$

**Proof.** We first provide the proof for  $O_k$ . It is straightforward to check by induction, and also well known [4, p. 239] that the Hamming distance along a shortest  $xy$ -path assumes the following values:

$$H_{xy} = 2k-2, 2, 2k-4, 4, 2k-6, 6, \dots, J \quad (3)$$

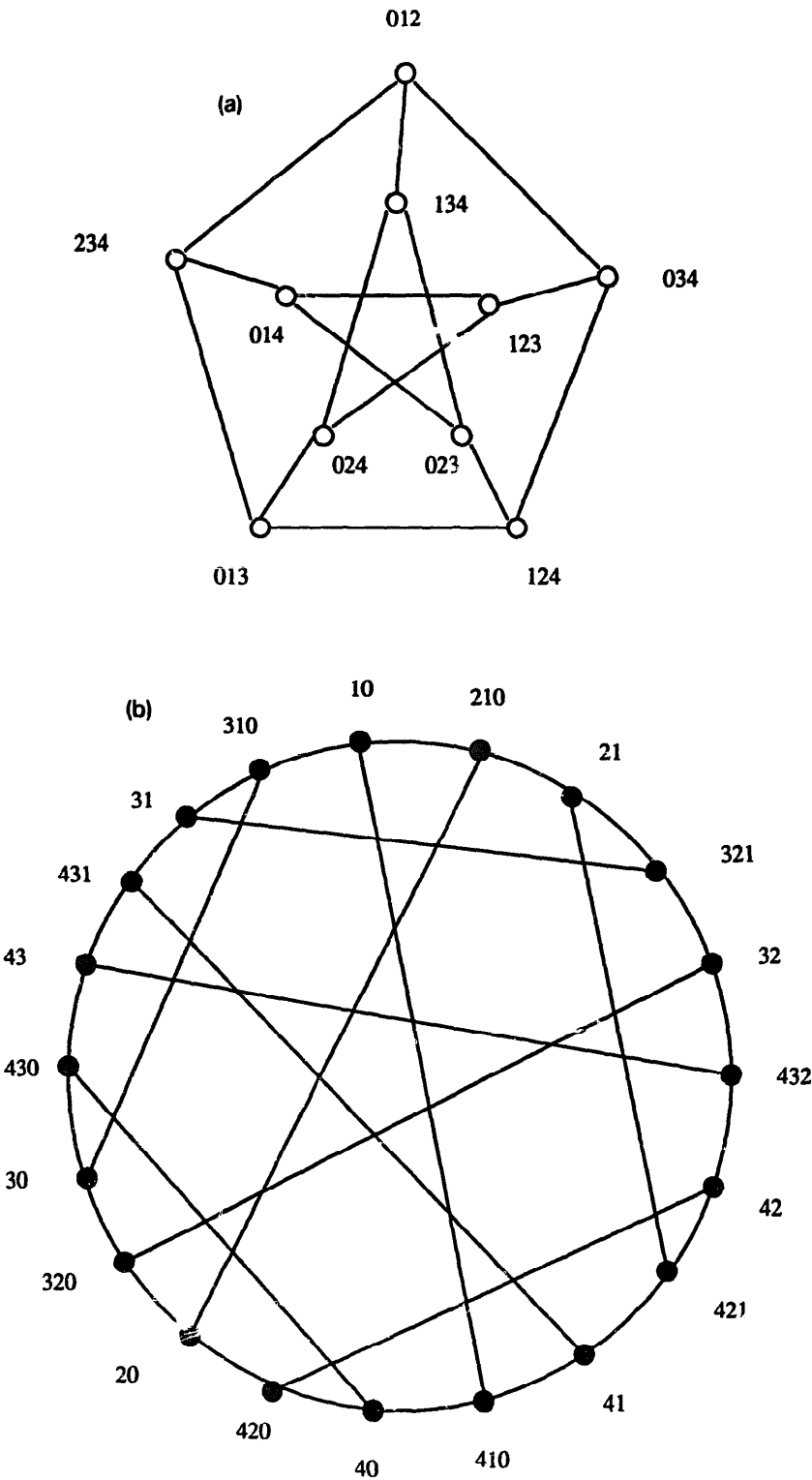


Fig. 1. (a)  $O_3$  graph,  $N=10$ ,  $k=3$ ,  $d=2$ . (b)  $2O_3$  graph,  $N=20$ ,  $k=3$ ,  $d=5$ .

with

$$H_{xy} = 1, 2k - 3, 3, 2k - 5, 5, \dots, 2k - 1 - J \quad (4)$$

while the graphical distance takes the values:

$$L_{xy} = 1, 2, 3, 4, 5, \dots, k - 1 \quad (5)$$

where

$$J = \begin{cases} k, & \text{if } d \text{ is even,} \\ k + 1, & \text{if } d \text{ is odd.} \end{cases}$$

From equations (3), (4) and (5) we note that for the even values,  $L_{xy} = H_{xy}$ , otherwise  $L_{xy} = H_{xy} - 1$  ( $= 2k - 1 - H_{xy}$ ). Furthermore, it is also obvious that for even distances from the root node  $H_{xy} < H_{xy}$ , while the reverse is true for odd distances. But  $H_{xy}$  is always even for constant weight codes [18] and as  $2k - 1$  is odd,  $H_{xy}$  is always odd. Therefore,  $L_{xy} = \min(H_{xy}, H_{xy})$ .

The proof for  $2O_k$  can be obtained similarly using induction and is quite straightforward.  $\square$

### 3. Fault tolerance and path enumeration of $O_k$ and $2O_k$ graphs

The fault-tolerance capability of an interconnection network is defined in terms of the minimum of all the openly disjoint paths (sometimes also called node disjoint) between pairs of nodes in the network. The set of openly disjoint path is also called a *container* [11,20]. According to Menger's theorem [6] the cardinality of such a set for a  $k$ -connected graph is the maximum and is equal to the degree  $k$ . This is true also for all edge-transitive graphs<sup>1</sup> [25], including both  $O_k$  and  $2O_k$  graphs [3,8]. Therefore, both these graphs are capable of maximal fault tolerance. For the enumeration of their openly disjoint paths in terms of length, we first need some definitions and a lemma, which are given below.

Given two nodes  $x$  and  $y$  in an  $O_k$  or  $2O_k$  graph, let  $A_{xy}$ ,  $B_{xy}$ ,  $C_{xy}$ , and  $D_{xy}$  be the sets defined *with respect to the subset associated with the vertex  $x$* , as follows:

$$\begin{aligned} A_{xy} &= x - y = \{\alpha_i\}, \quad i = 1, 2, \dots, |y - x|, \\ B_{xy} &= y - x = \{\beta_i\}, \quad i = 1, 2, \dots, |x - y|, \\ C_{xy} &= V - \{x \cup y\} = \{\gamma_i\}, \quad i = 1, 2, \dots, |V - \{x \cup y\}|, \\ D_{xy} &= \{x \cap y\} = \{\theta_i\}, \quad i = 1, 2, \dots, |\{x \cap y\}|. \end{aligned}$$

These sets, if not empty, are assumed to be arranged in some prescribed order, say in an ascending order. The elements (which we will term as *operators*)  $\alpha_i$ ,  $\beta_i$ ,

<sup>1</sup> Let  $\mathcal{A}$  be the automorphism group of a graph  $G$ .  $G$  is called *edge transitive*, if for each pair of edges  $\{x, y\}$  and  $\{u, v\}$  in  $G$ , there is some automorphism  $g$  in  $\mathcal{A}$ , satisfying  $g\{x, y\} = \{g(x), g(y)\} = \{u, v\}$  [25]. *Edge transitivity* is quite different from *vertex transitivity*, see [12] for detail.

$\gamma_i$  and  $\theta_i$  are the  $i$ th elements in these ordered sets, respectively. Any path between vertices  $x$  and  $y$  can be expressed in terms of these operators. For example, for nodes  $x = \{0, 1, 2, 3, 4, 5\}$  and  $y = \{0, 1, 2, 3, 6, 7\}$ , in  $O_6$ , we first notice that  $A_{xy} = \{4, 5\}$ ,  $B_{xy} = \{6, 7\}$ ,  $C_{xy} = \{8, 9, 10\}$ , and  $D_{xy} = \{0, 1, 2, 3\}$ . Accordingly, one possible shortest path between nodes  $x$  and  $y$  can consist of vertices  $\{0, 1, 2, 3, 4, 5\}$ ,  $\{4, 6, 7, 8, 9, 10\}$ ,  $\{0, 1, 2, 3, 5, 6\}$ ,  $\{4, 5, 7, 8, 9, 10\}$ , and  $\{0, 1, 2, 3, 6, 7\}$ . This path can be expressed from  $x$  as the sequence of operators  $\alpha_1 = 4$ ,  $\beta_1 = 6$ ,  $\alpha_2 = 5$ ,  $\beta_2 = 7$ . The selection of operators and the generation of paths is described in detail in the proof of Theorem 3.2.

The following lemma, which is a generalization of the discussion in [8], describes the structure of elementary cycles of even length, in terms of operators, introduced above.

**Lemma 3.1.** *In an  $O_{k \geq 4}$  graph, the concatenation of two sequences of operators, each consisting of  $\delta$  distinct operators, with one sequence being an arbitrary cyclic permutation of the other, constitutes an elementary cycle of even length  $2\delta$ .*

**Proof.** We first consider even length ( $\delta$ ) shortest-path sequences, between some arbitrary vertices  $x$  and  $y$ , with  $A_{xy}$  and  $B_{xy}$  being the sets associated with them, as defined above. Let  $P_1$  and  $P_2$  be the sets of sequences of operators, corresponding to two such even length paths, with  $P_2$  being some  $i$ th cyclically permuted version of  $P_1$ . Note, that being a shortest-path sequence, it consists of distinct operators. Also note, that once an operator  $\alpha_i \in A_{xy}$ , is applied to the  $k$ -subset, associated with the vertex  $x$ , it results in an inclusion only of the element  $\alpha_i$ , from the subset  $A_{xy}$ , while all the other elements in  $x$  are excluded as we move away from  $x$  by one edge, along the path described by the sequence of operators. The included element is then excluded while selecting the next vertex along that path. The selection of the next vertex is done through an operator, which with respect to the set  $x$  is of type  $\beta$  (see the example shown above), which results in an inclusion of the element  $\beta_i$  of the subset  $B_{xy}$ . Therefore, the element of  $x$ , which is retained through  $\alpha_i$ , goes through a sequence of exclusion and inclusion from subsets of vertices, as we move along the vertices present in the path under consideration. Therefore, if  $\alpha_j$  is the first operator in  $P_1$ , then the corresponding element in the first  $i$  vertices selected through the operators of the sequence of  $P_2$  must be different from that of the first  $i$  vertices selected through the sequence of operators of  $P_1$ . On the other hand,  $\alpha_j$  is the  $i$ th operator in  $P_1$ . Therefore, the element associated with this operator in the next  $i$  vertices corresponding to the operators of  $P_1$  must be different from the corresponding  $i$  vertices selected through the operators of  $P_2$ . A repeated use of this argument can lead to the conclusion that the elements corresponding to every  $g$ th operator ( $g = 1, i, (2i - 1), \dots, q < 2\delta$ ), if considered collectively, give rise to different  $k$ -subsets, for the vertices present at the same position of the two paths. Therefore, both the paths corresponding to operator sequences of  $P_1$  and  $P_2$  are openly disjoint. A little thought can reveal that the  $k$ -subset of the vertex, joining the operator

sequences of  $P_1$  and  $P_2$  at a given end, must have elements which correspond to those  $\alpha_g$  which are present at odd distances in both the paths, from the starting vertex  $x$ . Since both sequences correspond to the shortest paths, we have an elementary cycle of length  $2\delta$ .

The proof for odd length sequences can be provided similarly.  $\square$

The path enumeration theorem is given below.

**Theorem 3.2.** *The number of openly disjoint paths between any two nodes  $x, y$  of  $O_k$  and  $2O_k$  is equal to the degree  $k$ . The enumeration of these paths for both the graphs is given as follows:*

*Case (a):  $L_{xy}$  = even.*

*There are  $L_{xy}/2$  paths of length  $L_{xy}$ . The rest of the  $k - L_{xy}/2$  alternate paths are of length  $L_{xy} + 2$ . For  $O_k$ , in case  $L_{xy} = d$ , the diameter, the alternate paths are of length  $L_{xy} + 1$ .*

*Case (b):  $L_{xy}$  = odd.*

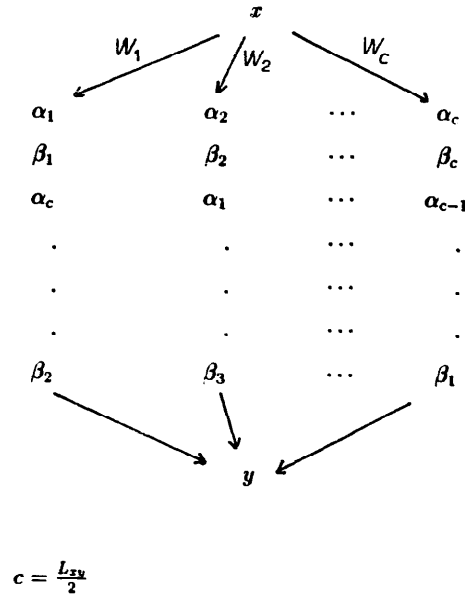
*There are  $(L_{xy} + 1)/2$  paths of length  $L_{xy}$  (this includes the case of  $L_{xy} = d$  for a  $2O_k$  graph). The rest of the  $k - (L_{xy} + 1)/2$  alternate paths are of length  $L_{xy} + 4$ . For  $O_k$ , in case  $L_{xy} = d$ , the alternate paths are of length  $L_{xy} + 1$ .*

*Furthermore, these disjoint paths include all the shortest possible paths between nodes  $x$  and  $y$ .*

**Proof.** Being edge transitive, both  $O_k$  and  $2O_k$ , have the maximum connectivity [25]. We enumerate the paths by actually constructing  $k$  openly disjoint paths between any pair of nodes.

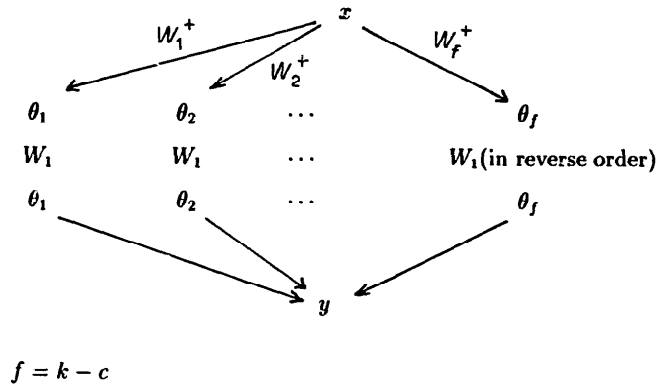
*Case 1:  $O_k$ .* For the case of  $L_{xy}$  being even, consider the paths  $W_1, W_2, \dots, W_{L_{xy}/2}$ , shown in Fig. 2 from vertex  $x$  to  $y$ . It can be noticed from the proof of the above lemma that these  $L_{xy}/2$  paths are openly disjoint and lead to node  $y$  from node  $x$ . The length of each such path is the minimum. This can be proved by noticing that the number of  $\alpha$  operators is  $L_{xy}/2$ . This results in a Hamming distance which is equal to  $L_{xy}$  between subsets  $x$  and  $y$ , by Theorem 2.2. Since all the possible cyclic permutations of  $\alpha$  and  $\beta$  operators are present in Fig. 2, the number of such shortest openly disjoint paths cannot exceed  $L_{xy}/2$ .

For the alternate paths of length  $L_{xy} + 2$  between vertices  $x$  and  $y$ , consider  $(L_{xy} + 1)/2$  paths  $W_1^+, W_2^+, \dots, W_{(L_{xy} + 1)/2}^+$ , each consisting of a sequence of operators of a path of Fig. 2, say  $W_1$ , taken in a reverse order, with two  $\theta$  operators appended at both of its ends, as shown in Fig. 3. Note, a path  $W_i^+$  differs from any other path  $W_j^+$ , with respect to the elements  $\theta_i$  and  $\theta_j$ . Therefore, all the paths shown in Fig. 3 are node disjoint. They lead to a single codeword as a direct consequence of Lemma 3.1, as a combination of both such paths yields a cycle of even length. These are the shortest possible alternate paths for all even values of  $L_{xy}$  ( $\neq d$ ), since an alternate path cannot be of length  $L_{xy}$ , as mentioned above. An alternate path of length  $L_{xy} + 1$  will result in an odd cycle of length  $2L_{xy} + 1$ .

Fig. 2. Direct path for  $L_{xy}$  = even in  $O_k$ .

This is not possible, because if  $L_{xy} = d - 1$ , it results in an odd cycle of  $2d - 1$  while the minimum odd cycle in  $O_{k \geq 4}$  is of length  $2d + 1$  [8]. The next choice is of length  $L_{xy} + 2$ , which is selected in Fig. 3. For  $L_{xy} = d$ , we can choose a single cycle of length  $2d + 1$  in order to select an alternate path. Note that Fig. 3 can also provide an alternate path even for the case of  $L_{xy} = d$ , although such a path may be slightly longer than the one described.

For the case when  $L_{xy}$  is odd, consider the paths  $Y_1, Y_2, \dots, Y_{(L_{xy}+1)/2}$ , from node  $x$  to node  $y$ , as shown in Fig. 4. Using the above argument, and observing the similarity between Figs. 4 and 2, it is noted that all these paths, except probably  $Y_1$  and  $Y_{(L_{xy}+1)/2}$ , are distinct. The latter paths can also be shown to be disjoint, by using a similar argument for the operators  $\gamma$  and  $\theta$ .

Fig. 3. Alternate path for  $L_{xy}$  = even in  $O_k$ .



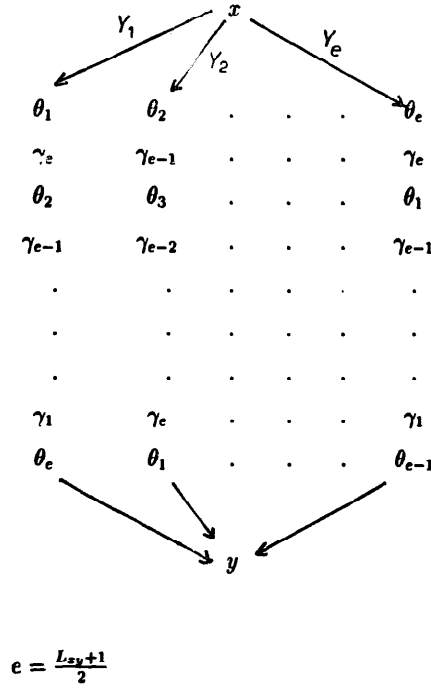


Fig. 4. Direct path for  $L_{xy} = \text{odd}$  in  $O_k$ .

The alternate paths of length  $L_{xy} + 4$  between nodes  $x$  and  $y$ , are shown as  $Y_1^+, Y_2^+, \dots, Y_{(L_{xy}+1)/2}^+$  in Fig. 5. Each of these paths consists of one of the paths of Fig. 4, say  $Y_1$ , taken in a reverse order, with operators  $\alpha$  and  $\beta$  appended at both ends, as shown in Fig. 5. As these operators only operate on the sets  $A_{xy}$  and  $B_{xy}$ , respectively, for which operators for the path  $Y_1$  has no connection, any two paths  $Y_i^+$  and  $Y_j^+$  differ for elements associated with  $\alpha_i, \beta_i$  and  $\alpha_j, \beta_j$  taken collectively. From the structure of these paths, we note that by combining  $Y_1$  with any  $Y_j^+$ , we get a cycle of even length, according to Lemma 3.1. Therefore, these paths

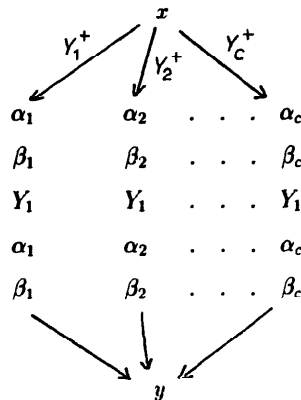
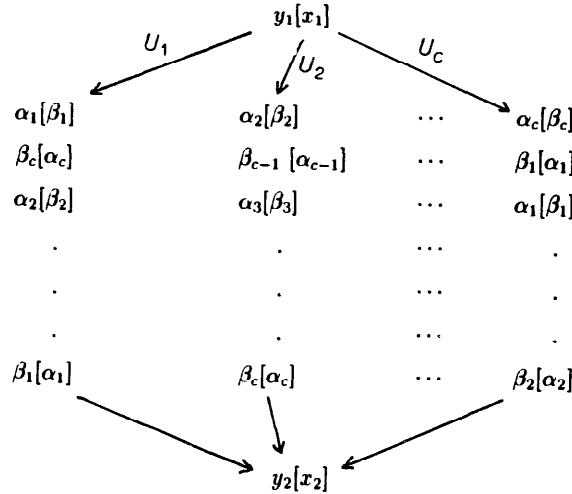


Fig. 5. Alternate path for  $L_{xy} = \text{odd}$  in  $O_k$ .

lead to the node  $y$ . Regarding the length  $L_{xy} + 4$  being the shortest for the alternate paths, we observe that for all the values of  $L_{xy}$  ( $\neq d$ ) an alternate path together with  $Y_1$ , must form an even cycle of length less than  $2d + 1$ . The even length is required because the  $k$ -subset of  $y$  contains elements of the sets  $D_{xy}$  and  $B_{xy}$ . The shortest path  $Y_1$  already consists of operators in  $D_{xy}$ . Therefore, in order to originate a new path from  $y$  we must select an operator from the set  $D_{xy}$ . This path must also be connected to the node  $x$ , through an operator belonging to the set  $A_{xy}$ , which is of the type  $\alpha$ . Therefore, by appending these two operators with path  $Y_1$  and using Lemma 3.1, the only choice to make a closed circuit is to have an alternate sequence of total length of  $L_{xy} + 4$ . Using the above line of argument for the case of  $L_{xy} = d$ , we can also find alternate paths of length  $k + 2$ .

Note that Fig. 5, can also provide an alternate path for the case of  $L_{xy} = d$ , which is slightly longer than the one described.

**Case 2:  $2O_k$ .** Again, for this graph we provide the proof by constructing and enumerating all such paths between the two vertices  $x$  and  $y$ . For  $L_{xy}$  even, consider the shortest and the alternate paths,  $U_i$  and  $U_i^+$ , as shown in Figs. 6 and 7, respectively. The number of paths can be verified from these figures. The proof that all these paths are openly disjoint and all the shortest paths of even lengths, say between two vertices having a  $k$ -subset,  $y_1$  and  $y_2$ , are included in Fig. 6, can be given using the same line of argument as for  $O_k$  graphs. This proof can also be extended to the case when vertices are  $(k - 1)$ -subsets.



$x_i$  is  $(k-1)$ -subset.

$y_i$  is  $k$ -subset.

$$|y_1 - y_2| = |y_2 - y_1| = \frac{L_{y_1 y_2}}{2} = c$$

$$|x_1 - x_2| = |x_2 - x_1| = \frac{L_{x_1 x_2}}{2} = c$$

Fig. 6. Direct path for  $L_{y_1 y_2} = \text{even}$  in  $2O_k$ .

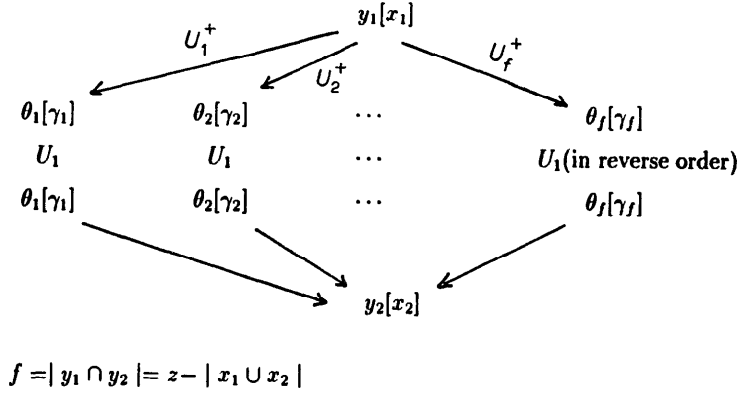


Fig. 7. Alternate path for  $L_{y_1, y_2} = \text{even}$  in  $2O_k$ .

The shortest and the alternate paths for odd  $L_{xy}$  are shown in Figs. 8 and 9, respectively. These are, respectively, labeled as  $R_i$  and  $R_i^+$ . Such paths exist between a pair of vertices with  $k$  and  $k-1$  subsets, respectively. The proof for these paths can be provided using the same line of argument as for the above cases.  $\square$

According to Theorem 3.2, both the  $O_k$  and  $2O_k$  graphs can withstand up to  $k-1$  failed nodes in the network. Therefore, these networks are capable of maximal fault tolerance.

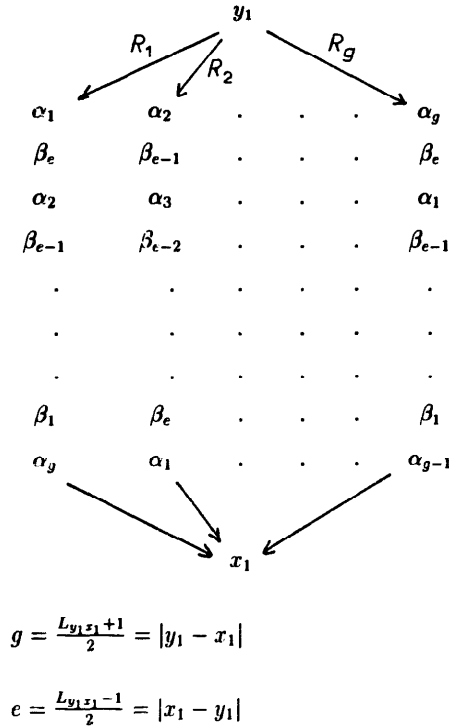
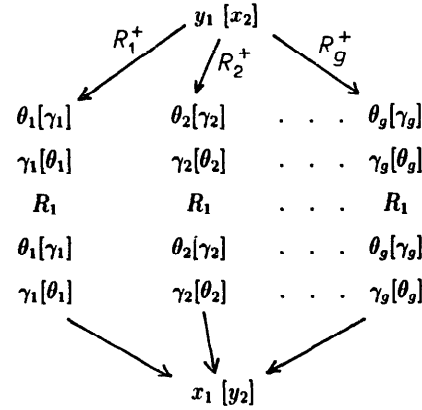


Fig. 8. Direct path for  $L_{y_1, x_1} = \text{odd}$  in  $2O_k$ .



$$g = |y_1 \cap x_1| = z - |y_1 \cup x_1|$$

Fig. 9. Alternate path for  $L_{y_1, x_1} = \text{odd}$  in  $2O_k$ .

#### 4. Maximum length of the container, persistence and resilience in $O_k$ and $2O_k$ graphs

Related to fault tolerance there are four important performance parameters which are called *fault diameter*, *persistence*, *resilience* and the *maximum length of the container* [1,9,20]. The detail of these parameters for the graphs under study and a comparison with other well-known networks is given in this section.

The first parameter of interest is the  $f$ -*fault diameter* ( $\Delta_f$ ) which is the maximum of the diameters for all possible subgraphs of a graph with  $f$  or fewer faulty nodes [20]. The most interesting case is when  $f = k - 1$ , which reflects the worst-case failure in the network, without causing disconnection of the network. For the graphs under study, we have the following theorem.

**Theorem 4.1.** *For an  $O_k$  graph  $\Delta_{k-1}$  is  $d+2$  and  $d+3$ , for  $k$  being even or odd, respectively. For the  $2O_k$  graph  $\Delta_{k-1} = d+2$ .*

**Proof.** For  $k$  even, it can be verified from Theorem 3.2 that the worst-case length occurs for the alternate path associated with  $L_{xy} = d - 2 = k - 3$ , for which the length of the alternate paths is  $d + 2$ , which is the value of  $\Delta_{k-1}$ . (Note that if we consider  $L_{xy} = d$ , then the length of the alternate paths is smaller and is  $d + 1$ .) Similarly, for  $k$  odd the worst-case length occurs for the largest odd value of  $L_{xy}$  which is  $d - 1 (= k - 2)$ . For this value of  $L_{xy}$ , we note from Theorem 3.2 that the length of the alternate paths is  $L_{xy} + 4$ , which results in  $\Delta_{k-1} = d + 3$ .

For the  $2O_k$  graph, we note that the worst case occurs for odd  $L_{xy} = d - 2$ , which results in the length of alternate paths being  $d + 2$ . For any other value of  $L_{xy}$ , we can verify from Theorem 3.2 that the length of the alternate paths is always less than  $d + 2$ .  $\square$

Graphs with  $\Delta_{k-1} = d + O(1)$  are called *strongly resilient* [1]. From Theorem 4.1, it is clear that both  $O_k$  and  $2O_k$  graphs are strongly resilient.

The third important fault-tolerance parameter is persistence [9]. A graph is called  $\mu$ -persistent, if  $\Delta_\mu = d$ . In other words, this parameter indicates the minimum number of nodes which can fail without increasing the diameter of the graph. The persistence of both  $O_k$  and  $2O_k$  graphs is given by the following lemma.

**Lemma 4.2.** *An  $O_k$  graph is  $(\lfloor k/2 \rfloor - 2)$ -persistent. A  $2O_k$  graph is  $(k - 2)$ -persistent.*

**Proof.** According to Theorem 3.2, we note that for an  $O_k$  graph the maximum value of  $L_{xy}$  for which the length of the alternate path does not exceed the diameter  $d$ , is given by  $d - 2 = k - 3$  when  $d$  is even, otherwise it is  $d - 4 = k - 5$ . Accordingly, the number of shortest path of such a length is  $L_{xy}/2$  and  $(L_{xy} + 1)/2$ , respectively. This results in the stated value, that is, a failure of up to  $(\lfloor k/2 \rfloor - 2)$  nodes in the  $O_k$  graph, does not cause the diameter of the resulting subgraph to exceed  $d$ .

For the  $2O_k$  case, again referring back to Theorem 3.2, we note that the maximum value of  $L_{xy}$  for which the length of the alternate path does not exceed the diameter  $d$ , is odd and is given by  $d - 4 = 2k - 5$ . Accordingly, the number of shortest path of such a length is  $(L_{xy} + 1)/2$ , which results in the stated value  $k - 2$ . Accordingly, a failure of up to  $k - 2$  nodes in the  $2O_k$  graph, does not cause the diameter of the resulting subgraph to exceed  $d$ .  $\square$

The fourth parameter of interest is the maximum of the lengths of the “best” containers with  $f$  openly disjoint, taken over all possible pairs of nodes in a graph (here the length of the container implies the length of the longest path in the container and the “best” refers to the containers having the best possible paths in terms of length) [20]. This parameter is denoted as  $\Delta_{[f]}$ . The most interesting case is for  $f = k$ . Again, from Theorem 3.2, we have the following lemma.

**Lemma 4.3.** *For both  $O_k$  and  $2O_k$ ,  $\Delta_{[k]} = \Delta_{k-1}$ .*

The above lemma is important, since it leads to the conclusion that these graphs are as good as flip-tree networks, which have the best known value for  $\Delta_{[k]}$  [20]. We now summarize all the observations about  $O_k$  and  $2O_k$  graphs, and compare them with various well-known networks.

Table 1 contains such a summary of various graphs. When  $k$  and  $d$  are not related, their values can be arbitrary, shown as  $x$  and  $y$ . As a result, the number of nodes  $N$  in a graph is expressed as the function of  $x$  and  $y$ . Similarly, the appropriate fault-tolerance parameters are expressed in terms of  $x$  and  $y$ , in Table 1. For comparison, we first consider the  $(k - 1)$ -fault diameter,  $\Delta_{k-1}$ . For de Bruijn networks this value is  $\infty$  [20]. This makes both  $O_k$  and  $2O_k$  graphs superior to de Bruijn graphs in terms of  $(k - 1)$ -fault diameter, although the latter is relatively more

Table 1.  
Comparison of fault diameter of various topologies

Graph	$N$	$k$	$d$	$\Delta_{k-1}$	$\Delta_{[k]}$	Persistence
Odd graphs	$\binom{2x-1}{x}$	$x$	$x-1$	$x+1$ ( $x$ even), $x+3$ ( $x$ odd)	$\Delta_{k-1}$	$\left\lfloor \frac{x}{2} \right\rfloor - 2$
$2O_k$	$\binom{2x}{x}$	$x$	$2x-1$	$2x+1$	$\Delta_{k-1}$	$x-2$
Flip-tree	$\frac{y(y-1)^x-2}{y-2}$	$y$	$2x-1$	$2x+1$	$\Delta_{k-1}$	0
Star	$x!$	$x-1$	$\left\lceil \frac{3x-4}{2} \right\rceil$	$\left\lceil \frac{3x+1}{2} \right\rceil$	?	?
De Bruijn	$y^x$	$2y$	$x$	$\infty$	$\infty$	0
Generalized hypercube	$y^x$	$(y-1)x$	$x$	$x+1$	$\Delta_{k-1}$	$\left\lfloor \frac{x}{2} \right\rfloor$
Cube-connected cycle (CCC)	$x2^x$	3	$\left\lceil \frac{5x-5}{2} \right\rceil$	$\left\lceil \frac{5x+1}{2} \right\rceil$	$\Delta_{k-1}$	0

dense. We also note, that both  $O_k$  and  $2O_k$  graphs are equally good as compared to other networks in terms of this parameter.

Next we note, that  $O_k$  and  $2O_k$  graphs are as good as any other network with respect to the length of the best container,  $\Delta_{[k]}$ . For de Bruijn graphs this parameter has an  $\infty$  value, while for star graphs its value is unknown (although the values given in [1] make them rather inferior as compared to the  $O_k$  and  $2O_k$  graphs).

Regarding persistence, we note that  $O_k$  and  $2O_k$  graphs are comparatively better than most of the other networks, such as de Bruijn, flip-trees etc., since the latter have no persistence [20].

Both  $O_k$  and  $2O_k$  graphs, allow very simple routing algorithms both for faulty and nonfaulty networks. In this paper we have concentrated on the fault-tolerance properties of these networks. The discussion about routing algorithms for faulty and nonfaulty  $O_k$  and  $2O_k$  networks can be found in [13–15].

In summary, we conclude that from the point of view of fault tolerance, resilience, persistence, and length of the best containers both  $O_k$  and  $2O_k$  graphs inherit all these properties as compared to many other known architectures which possess a subset of them.

## 5. Partitioning of $O_k$ and $2O_k$ networks for improved diagnosability

An efficient distributed diagnosis algorithm should have as small overhead (such as requirement for the software or hardware, or the amount of traffic generated

etc.) as possible. It should also be independent of any graphical consideration. The former requirement can be met if the algorithm is semi-distributed, that is, a small number of nodes are used to implement the algorithm. For the latter requirement, the nodes selected for running the algorithm must be located in the network with some form of symmetry. For symmetry, we present the important property of  $O_k$  and  $2O_k$  networks which is their capability of partitioning into identical spheres. For this, we introduce the notion of *uniform set of medians* in a network and identify spheres, centered at the medians. As mentioned earlier, this partitioning property can be exploited to increase the overall diagnosability of  $O_k$  and  $2O_k$  networks well beyond the previously obtained value of diagnosability which is equal to the degree [13,14].

### 5.1. Uniform set of medians

A *uniform set of medians* is the maximal set of nodes such that the graphical distance between any two nodes is equal to the diameter of the network.

The problem of finding such a maximal set, that is a set in which every node is at a distance of at least some fixed value (say  $\delta$ ) from every other node in the set, is also known as *maximum  $\delta$ -separated matching* and is NP-complete for an arbitrary graph, for  $\delta > 2$  [24]. However, for  $O_k$  graphs, finding such a maximal set (for  $\delta = k - 1$ ) is a trivial problem as described below. For the  $2O_k$  graph, which is an anti-podal graph, we take the set from  $O_k$ , and its complement (see the discussion below).

Suppose there exists a Hadamard matrix  $M$ , which is a  $j \times j$  matrix with  $\pm 1$  entries, such that  $MM^T = jI$ , where  $I$  is the identity matrix and  $M^T$  is the transpose of  $M$ . If we replace 1 by 0, and  $-1$  by 1, we shall say the matrix is in 0-1 notation. Let a set  $\Gamma$  consist of the nodes having codewords as rows of  $M$ . It is known that for all the values  $v + 1 \leq 268$ , where  $v + 1$  is a multiple of 4, the desired Hadamard matrices do exist [18]. Furthermore, it has been conjectured that a Hadamard matrix of any order, which is a multiple of 4, does exist [18]. It can be noted that such values of  $v$  represent even values for degree  $k$ , for  $O_k$  graphs. For the case when  $v$  is not a multiple of 4 (which is the case of odd value of  $k$ ), but it is even, the selection of the set  $\Gamma$  can be done as follows:

Take the set  $\Gamma$  for the case of  $v - 1$ , for which a Hadamard matrix does exist. By appending two bits 01 at any two fixed positions, say at extreme left, of the codewords associated with the members of the set  $\Gamma$ , we generate, the set  $\Gamma^+$ . The length of the new codewords is  $v$  and their weight is  $k + 1$ . Clearly, the elements of  $\Gamma^+$  are in the graph  $O_{k+1}$ .

For the  $2O_k$  graph, the set  $\Gamma$  is selected, by taking rows of both truncated matrix  $M$  (in 0-1 notation) and its complementary matrix  $M^c$ . It is obvious, the set  $\Gamma$  in this case constitutes a self-complementary code.

We now prove that the set  $\Gamma$  provides the maximal  $k - 1$ -separated matching for an odd graph:

**Theorem 5.1.** *Given  $x, y \in \Gamma$ ,  $L_{xy} = k - 1$ , and it is the maximal possible set, with  $k$ -separated matching.*

**Proof.** In a normalized truncated form, the Hamming distance between any two rows of a Hadamard matrix  $M$  is  $(v+1)/2$  [18]. Since  $v = 2k - 1$ , therefore, according to Theorem 2.2, the graphical distance between any two nodes with addresses as the rows of  $M$  must be equal to  $k - 1$ . In order to prove that the cardinality of the set is the maximum possible one, let us assume the contrary is true, and suppose there exists some codeword  $z$ , such that  $H_{zx} = (v+1)/2$ ,  $\forall x \in \Gamma$ . A simple counting argument can reveal, that there must be a total of  $(v+1)(v-1)/4$  1's at those  $k-1$  columns where  $z$  has 0's. If we distribute these 1's among all the rows of  $M$  to have the desired Hamming distance of  $(v+1)/2$  between  $z$  and *all* the rows of  $M$ , we can immediately find that there are at least  $v - (v+1)(v-1)/(4v)$  rows which cannot be filled to obtain this Hamming distance. Therefore, according to Theorem 2.2, the node  $z$  will be at a graphical distance less than  $k - 1$  from these rows. Since we are assuming that  $v$  is a multiple of 4,  $k$  must be even. The proof for the case of  $k$  odd, can be provided using the same line of argument.  $\square$

**Lemma 5.2.** *Since, for the  $2O_k$  graph,  $\Gamma$  is a self-complementary code,  $L_{xy}$  is either  $2k - 1$ ,  $k - 1$  or  $k$ .*

The set of seven medians for the  $O_4$  network is shown in Fig. 10. The same set, after appending two 01 bits, generate the set  $\Gamma^+$  for the  $O_5$  graph (see Fig. 11).

The generation of the Hadamard matrix using SBIBD's is a trivial process. Since most of the desired SBIBD's are cyclic in nature, that is all the blocks (which correspond to the elements of the set  $\Gamma$ ) can be generated by taking  $2k - 1$  cyclic shifts of a single generator codeword. Such generators, for different values of  $k$  can be found in a straightforward manner, using the so-called *difference set approach* [5,18]. Figure 12 lists generator codewords for various values of  $k$ .

	0	0	1	0	1	1	1
	0	1	0	1	1	1	0
	1	0	1	1	1	0	0
$M =$	0	1	1	1	0	0	1
	1	1	1	0	0	1	0
	1	1	0	0	1	0	1
	1	0	0	1	0	1	1

Fig. 10. A truncated Hadamard matrix of order 7 (the set  $\Gamma$  for  $O_4$ ).



0	1	0	0	1	0	1	1	1
0	1	0	1	0	1	1	1	0
0	1	1	0	1	1	1	0	0
0	1	0	1	1	1	0	0	1
0	1	1	1	1	0	0	1	0
0	1	1	1	0	0	1	0	1
0	1	1	0	0	1	0	1	1

Fig. 11. Set of seven medians for the  $O_5$  network.

### 5.2. Network partitioning and diagnosability

We now present a strategy which can increase the overall diagnosability of  $O_k$  and  $2O_k$  graphs well beyond  $k$  and  $2k-1$ , respectively. This strategy utilizes the remarkable partitioning property of these graphs. This property is based on the use of the codewords of  $\Gamma$  which divide the  $O_k$  graph into  $2k-1$  symmetrical, nonoverlapping regions, if  $k$  is even. Similarly, for  $2O_k$  graphs we have  $2k$  regions. The number of such regions is  $2k-3$  for  $O_k$  graphs and  $4k-6$  for  $2O_k$  graphs, if  $k$  is odd. Employing a semi-distributed fault-diagnosis procedure in each region and using the test results on a global basis, we can achieve an overall diagnosability of the order of  $\theta|\Gamma|$ , for  $\theta > 1$ .

The partitioning of  $O_k$  is based upon choosing a set of nodes, which can serve as the centers for the partitioned regions and distributing the rest of nodes in the network among these regions. Clearly, a natural distribution is to cluster the neighboring nodes around the selected centers. For the proposed scheme we select nodes of the set  $\Gamma$  as these centers. We distribute nodes among the regions almost evenly. For example, for  $O_6$  we have a total of 462 nodes which are distributed among 11 regions. Therefore, each region contains 42 nodes. The association of a node in the network to its nearest center (which we will call algorithmic nodes) can be determined statically at the time the system is designed. Once the membership of each region is determined, each region can be treated as an independent connected subnetwork, in order to implement the proposed fault-diagnosis mechanism. The interregional

$k$	Generator Row
5	0 0 1 0 1 1 1
7	1 0 1 1 1 0 0 0 1 0 1
9	1 1 1 1 0 1 0 1 1 0 0 1 0 0 0 0
11	1 0 0 1 1 1 1 0 1 0 1 0 0 0 0 1 1 0 1
13	1 1 1 1 0 1 0 1 1 0 0 1 1 0 0 1 0 1 0 0 0 0 1

Fig. 12. Median generator rows.

links are excluded while identifying the regional subnetwork. This subnetwork can be modeled as a graph having some minimum connectivity  $\theta$ .  $\theta$  plays a vital role in determining the overall system's diagnosability. We, therefore, first describe a relationship between  $\theta$  and  $k$ , and then use it to determine the diagnosability. This relationship depends upon the covering radius  $r$  of the Hadamard code in  $O_k$  which is given by the following lemma [23].

**Lemma 5.3.** *For  $O_k$  graphs, an asymptotic lower bound on  $r$  is:*

$$r \geq k - c\sqrt{k \log k} + O(\sqrt{k}), \quad (6)$$

where  $c = 0.833$ .

Bounds like equation (6) may seem quite crude, but they are sufficient for our purposes.

**Theorem 5.4.** *Let  $r$  be the covering radius of the Hadamard code in  $O_k$  (this radius is the same for  $2O_k$ , since we use the complementary code for this graph). Then  $\theta = \lceil r/2 \rceil$ .*

**Proof.** Note that  $\theta$  is determined by the upward connectivity of the nodes lying at the bottom of the regional tree. This is due to the fact that these nodes do not have any downward connectivity as such connections represent those links which lead to neighboring regions. Let  $\sigma_i$  and  $\zeta_i$  be the upward and downward connectivities, respectively, of a node lying at level  $i$ . Let the number of nodes at level  $i$  be  $v_i$ , the valency of  $O_k$  [13]. The following relation must hold: Using the expression of  $v_i$  from [13], we have

$$v_i \sigma_i = v_{i-1} \zeta_{i-1}$$

with  $\zeta_0 = 1$  and  $\sigma_1 = 1$ .

After some simplification, we get

$$\sigma_i = \left\lceil \frac{i}{2} \right\rceil. \quad (7)$$

Since the cardinality of  $\Gamma$  is  $2k-1$ , using the regional covering argument in  $O_k$ , we get the following, for even values of  $k$ :

$$\sum_{i=0}^r w_i \geq \frac{\binom{2k-1}{k}}{2k-1} \quad (8)$$

where the  $w_i$  are given by the following valency sequence of an  $O_k$  graph:

$$v_0, v_{2k-2}, v_1, v_{2k-4}, \dots$$

The use of equation (8), gives the desired result. The case for odd values of  $k$  and  $2O_k$  can be proved on the same lines.  $\square$

We now discuss the diagnosability of the partitioned network and propose a diagnostic algorithm.

### 5.3. *Diagnosability and selection of algorithmic nodes in partitioned networks*

As each region has a minimum connectivity  $\theta$ , it is noted [21] that the diagnosability of each region is  $\theta$ . In order to achieve this diagnosability, we now propose a semi-distributed algorithm for each region which can help in identifying  $\theta$  faulty nodes. This algorithm is executed by  $\theta + 1$  regional algorithmic nodes, which are responsible for running their local diagnostic algorithm. In order to select these regional algorithmic nodes, the only major consideration which we need to take is that these nodes should not be neighbors of each other, otherwise failure among the diagnostic nodes may go undetected. Keeping this consideration in mind, we now give a rule for selecting these regional algorithmic nodes.

For each region, the median node from the set  $\Gamma$  can be selected as one of the algorithmic nodes. It is shown in [14] that for  $O_k$  graphs in any region, we can easily find at least  $k(k-1)/2 + 1$  nodes, for  $k$  greater than 8, such that any two nodes are at least at a distance of 3 apart. For  $d=6$  or 7, a set of  $k+1$  number of such nodes is possible. The same situation is also possible for  $2O_k$  graphs.

### 5.4. *Diagnostic algorithm for the partitioned networks*

Since each region has a minimum connectivity  $\theta$ , we now present a semi-distributed diagnosis algorithm. In each region we select  $\theta + 1$  number of nodes, which are responsible for executing the algorithm and identifying up to  $\theta$  faulty nodes within their respective region.

During the diagnostic phase of the system, the algorithmic nodes in each region initiate test procedures, using their respective regional trees rooted at algorithmic nodes. All regions simultaneously conduct their tests in the form of rounds, and complete the results. The results are then exchanged among the regions, as each region still needs to know the faulty units in other regions. The algorithm is given as follows:

#### **Diagnosis algorithm.**

*Step 1.* Each algorithmic node initiates testing topdown on its respective regional tree.

*Step 2.* The test results are transmitted bottom up on each of the regional trees associated with the regional algorithmic nodes. Every node tests all its neighbors, including the ones lying in different regions. The test results about a neighbor in a different region need not be transmitted to an algorithmic node, rather such information is used by the testing node for the purpose of making the decision, whether it should exchange regional results or not, with the tested neighbor.

*Step 3.* A node at any level of such a tree neither passes results to, nor accepts results from, a neighbor if it has tested that neighbor and found it to be faulty. This includes the algorithmic nodes as well.

*Step 4.* Each algorithmic node examines the test results to identify all the faulty nodes in its region. If there are any faulty nodes, the root node broadcasts a message to the whole network, by sending it to all its nonfaulty neighbors. In the case that no faulty node is identified, no action is taken by the algorithmic node.

*Step 5.* A node in the network accepts the message about the test results (Step 4), from its neighbor only if it has tested that neighbor and found it to be fault free. The same criterion applies for a neighbor which lies in a different region.

*Step 6.* A node in the network, after receiving a message from a neighbor, sends it to all those neighbors whom it has tested and found to be fault free. Any neighbor in a different region is also included for this action.

In order to stop circulating such a message in the network forever, we can employ strategies such as time stamp, hop count or some sort of label for the round of tests.

**Theorem 5.5.** *The above algorithm diagnoses up to  $\theta(2k-1)$  faulty nodes in  $O_{k=\text{even}}$  and  $\theta(2k-3)$  faulty nodes in  $O_{k=\text{odd}}$ . For  $2O_k$  graphs, these numbers are doubled.*

**Proof.** As each region has minimum connectivity  $\theta$  and there are  $2k-1$  regions in the partitioned  $O_{k=\text{even}}$  network, the overall diagnosability is clearly  $\theta(2k-1)$  provided each node receives correct information about the results, which in turn, requires that only correct information is passed across the boundaries of each region. This is ensured by Steps 5 and 6.  $\square$

The above algorithm is semi-distributed and is executed by a very small set of nodes. We note that it provides remarkable improvement over the previous algorithms [13,14] in terms of the diagnosability of the system, since those algorithms can diagnose a number of nodes given only by the degree of the graph. The scheme proposed in this paper is much superior. For example, for an  $O_{k=\text{even}}$  graph, the proposed scheme can diagnose almost  $k^2$  nodes (according to Lemma 5.3 and Theorems 5.4 and 5.5), instead of  $k$ , which is a substantial improvement.

## 6. Conclusion

In this paper various fault-tolerant properties of both the  $O_k$  graph and its derivative  $2O_k$  have been analyzed. We provided a complete enumeration of all the node-disjoint paths between any pairs of nodes in these graphs. This enumeration proved maximal fault-tolerance capability of these graphs. Also, it provided a derivation of other important fault-tolerant performance parameters such as *f-fault diameter*, *persistence* and *resilience*.

Also, for these networks a semi-distributed diagnosis scheme has been presented. The scheme uses the partitioning capability of these graphs which is based on a

combinatorial structure, known as *Hadamard matrix*. The new scheme has been shown to provide an improvement in diagnosability of these networks which is of order of magnitude reported earlier [13,14]. These properties revealed that both these graphs are comparable and even better than many well-known dense graphs.

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